Journal of Approximation Theory **111**, 315–328 (2001)
doi:10.1006/jath.2001.3579, available online at http://www.idealibrary.com on IDEAL®

On the Semi-norm of Radial Basis Function Interpolants

Hans-Martin Gutmann

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom
E-mail: H.M.Gutmann@damtp.cam.ac.uk

Communicated by Robert Schaback

Received May 23, 2000; accepted in revised form February 20, 2001;
published online June 20, 2001

Radial basis function interpolation has attracted a lot of interest in recent years. For popular choices, for example thin plate splines, this problem has a variational formulation, i.e. the interpolant minimizes a semi-norm on a certain space of radial functions. This gives rise to a function space, called the native space. Every function in this space has the property that the semi-norm of an arbitrary interpolant to this function is uniformly bounded. In applications it is of interest whether a sufficiently smooth function belongs to the native space. In this paper we give sufficient conditions on the differentiability of a function with compact support, in the case of cubic, linear and thin plate splines. In the case of multiquadrics and Gaussian functions, it is shown that the only compactly supported function that satisfies these conditions is identically zero. © 2001 Academic Press

1. INTRODUCTION

Let n pairwise different points $x_1, ..., x_n \in \mathbb{R}^d$ be given, where n and d are any positive integers. Further, let $\lambda_1, ..., \lambda_n$ be real numbers, p be in Π_m, the space of polynomials of degree at most m, and denote the Euclidean norm by $\|.\|$. A radial basis function is of the form

$$s(x) = \sum_{i=1}^{n} \lambda_i \phi(\|x - x_i\|) + p(x), \qquad x \in \mathbb{R}^d. \tag{1.1}$$

In this paper we consider the following choices of ϕ:

$$\left.\begin{array}{ll} \phi(r) = r & \text{(linear)}, \\ \phi(r) = r^3 & \text{(cubic)}, \\ \phi(r) = r^2 \log\, r & \text{(thin plate spline)}, \\ \phi(r) = \sqrt{r^2 + c^2} & \text{(multiquadric)}, \\ \phi(r) = e^{-r^2} & \text{(Gaussian)}, \end{array}\right\} r \geqslant 0. \tag{1.2}$$

Let the matrix $\Phi \in \mathbb{R}^{n \times n}$ be defined by

$$(\Phi)_{ij} := \phi(\|x_i - x_j\|), \qquad i, j = 1, ..., n. \tag{1.3}$$

In all the cases in (1.2), Φ is conditionally definite. Specifically, letting m_0 be 1 in the cubic and thin plate spline case, 0 in the linear and multiquadric case and -1 in the Gaussian case, we obtain (see e.g. [5])

$$(-1)^{m_0+1} \lambda^T \Phi \lambda > 0, \tag{1.4}$$

if $\lambda \in \mathbb{R}^n$ is any nonzero vector that satisfies

$$\sum_{i=1}^{n} \lambda_i q(x_i) = 0 \qquad \forall q \in \Pi_m, \tag{1.5}$$

where Π_{-1} is the space that contains only the zero function. We choose m to be an integer that is not less than m_0.

Now fix $n \in \mathbb{N}$ and the centres $x_1, ..., x_n \in \mathbb{R}^d$. The interpolation problem can be posed in the following way. Find a radial basis function s of the form (1.1) that satisfies

$$\begin{aligned} s(x_i) &= f_i, \qquad && i = 1, ..., n \\ \sum_{i=1}^{n} \lambda_i q(x_i) &= 0, && q \in \Pi_m \end{aligned} \tag{1.6}$$

Let $\hat{m}$ be the dimension of Π_m. Further, let $p_1, ..., p_{\hat{m}}$ be a basis of this linear space. The matrix P is defined by

$$P := \begin{pmatrix} p_1(x_1) & \cdots & p_{\hat{m}}(x_1) \\ \vdots & & \vdots \\ p_1(x_n) & \cdots & p_{\hat{m}}(x_n) \end{pmatrix}.$$

In the Gaussian case with $m = -1$, P is omitted. Further, let $F \in \mathbb{R}^n$ be the vector whose entries are the data values $f_1, ..., f_n$. Therefore the system (1.6) can be written as

$$\begin{pmatrix} \Phi & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ c \end{pmatrix} = \begin{pmatrix} F \\ 0_{\hat{m}} \end{pmatrix} \tag{1.7}$$

where $\lambda \in \mathbb{R}^n$ has the components λ_i, where $c \in \mathbb{R}^{\hat{m}}$ and where $0_{\hat{m}}$ is the zero in $\mathbb{R}^{\hat{m}}$. The components of c are the coefficients of the polynomial p with respect to the basis $p_1, ..., p_{\hat{m}}$. Powell [5] shows that the interpolation matrix

$$A = \begin{pmatrix} \Phi & P \\ P^T & 0 \end{pmatrix} \in \mathbb{R}^{(n+\hat{m}) \times (n+\hat{m})} \tag{1.8}$$

is nonsingular for any prescribed $m \geqslant m_0$, if and only if $x_1, ..., x_n$ satisfy

$$q \in \Pi_m \quad \text{and} \quad q(x_i) = 0, \quad i = 1, ..., n \quad \Rightarrow \quad q \equiv 0. \tag{1.9}$$

An interesting and useful property is that the radial basis function that solves the interpolation problem (1.6) is the solution of a minimization problem. For any choice of ϕ in (1.2) and $m \geqslant m_0$, let us define the linear space $\mathscr{A}_{\phi, m}$ as the space of functions of the form

$$\sum_{i=1}^{N} \lambda_i \phi(\|x - y_i\|) + p(x), \qquad x \in \mathbb{R}^d,$$

where N is any positive integer, where $y_1, ..., y_N$ are any pairwise different points in $\mathbb{R}^d$, where p is any polynomial in Π_m and where $\lambda = (\lambda_1, ..., \lambda_N)^T$ satisfies

$$\sum_{i=1}^{N} \lambda_i q(y_i) = 0 \qquad \forall q \in \Pi_m. \tag{1.10}$$

On this space a semi-inner product and a semi-norm can be defined. Let s and u be any functions in $\mathscr{A}_{\phi, m}$, i.e.

$$s(x) = \sum_{i=1}^{N(s)} \lambda_i \phi(\|x - y_i\|) + p(x) \qquad \text{and} \qquad u(x) = \sum_{j=1}^{N(u)} \mu_j \phi(\|x - z_j\|) + q(x).$$

The semi-inner product is the expression

$$\langle s, u \rangle := (-1)^{m_0 + 1} \sum_{i=1}^{N(s)} \lambda_i u(y_i). \tag{1.11}$$

Clearly, it is bilinear. To show symmetry, by employing

$$\sum_{i=1}^{N(s)} \lambda_i q(y_i) = 0 \qquad \text{and} \qquad \sum_{j=1}^{N(u)} \mu_j p(z_j) = 0,$$

we deduce

$$\begin{aligned}\langle s, u\rangle &= (-1)^{m_0+1} \sum_{i=1}^{N(s)} \lambda_i \left(\sum_{j=1}^{N(u)} \mu_j \phi(\|y_i - z_j\|) + q(y_i) \right) \\ &= (-1)^{m_0+1} \sum_{i=1}^{N(s)} \sum_{j=1}^{N(u)} \lambda_i \mu_j \phi(\|y_i - z_j\|) \\ &= (-1)^{m_0+1} \sum_{j=1}^{N(u)} \mu_j \left(\sum_{i=1}^{N(s)} \lambda_i \phi(\|z_j - y_i\|) + p(z_j) \right) \\ &= (-1)^{m_0+1} \sum_{j=1}^{N(u)} \mu_j s(z_j) = \langle u, s\rangle.\end{aligned}$$

By (1.4), the expression

$$\langle s, s\rangle = (-1)^{m_0+1} \sum_{i=1}^{N(s)} \lambda_i s(y_i) = (-1)^{m_0+1} \sum_{i,\, j=1}^{N(s)} \lambda_i \lambda_j \phi(\|y_i - y_j\|) \tag{1.12}$$

is strictly positive, if λ satisfies (1.5) for $n = N(s)$ and any $m \geqslant m_0$. Thus (1.11) is indeed a semi-inner product. Then $(\langle s, s\rangle)^{1/2}$ is a semi-norm on the space $\mathcal{A}_{\phi, m}$ with null space Π_m.

Schaback [7] shows that the solution of (1.6) can be characterized as follows.

THEOREM 1. *Let ϕ be any function from* (1.2), *and let m be chosen such that $m \geqslant m_0$. Given are points $x_1, ..., x_n$ in $\mathbb{R}^d$ that satisfy* (1.9) *and values $f_1, ..., f_n$ in $\mathbb{R}$. Let s be the radial function of the form* (1.1) *that solves the system* (1.6). *Then s minimizes the semi-norm $\langle g, g\rangle^{1/2}$ on the set of functions $g \in \mathcal{A}_{\phi, m}$ that satisfy*

$$g(x_i) = f_i, \qquad i = 1, ..., n. \tag{1.13}$$

Proof. Let $g(x) := \sum_{j=1}^{n(g)} \mu_j \phi(\|x - y_j\|) + q(x) \in \mathcal{A}_{\phi, m}$ satisfy (1.13). We consider the semi-norm of $g - s$. Since $s(x_i) = g(x_i)$, $i = 1, ..., n$, it has the value

$$\begin{aligned}\langle g - s, g - s\rangle &= \langle g, g\rangle - 2\langle g, s\rangle + \langle s, s\rangle \\ &= \langle g, g\rangle + \langle s, s\rangle - 2(-1)^{m_0+1} \sum_{i=1}^{n} \lambda_i g(x_i) \\ &= \langle g, g\rangle + \langle s, s\rangle - 2(-1)^{m_0+1} \sum_{i=1}^{n} \lambda_i s(x_i) \\ &= \langle g, g\rangle + \langle s, s\rangle - 2\langle s, s\rangle \\ &= \langle g, g\rangle - \langle s, s\rangle.\end{aligned} \tag{1.14}$$

Thus, we deduce the required condition

$$\langle s, s\rangle = \langle g, g\rangle - \langle g-s, g-s\rangle \leqslant \langle g, g\rangle. \quad \blacksquare$$

One can associate with a particular ϕ in (1.2) and $m \geqslant m_0$ an interesting function space, the so-called native space $\mathcal{N}_{\phi, m}$ (see Schaback [8]). Schaback and Wendland [9] show that it can be characterized as the space of all functions F: $\mathbb{R}^d \to \mathbb{R}$ for which the following condition holds.

Condition 2. There exists a real number C, that only depends on F, such that, for any choice of interpolation points $x_1, ..., x_n \in \mathbb{R}^d$ for which (1.9) holds, the interpolant s_n to F at these points satisfies

$$\langle s_n, s_n\rangle \leqslant C.$$

For many types of radial basis functions, it is unknown what functions F belong to $\mathcal{N}_{\phi, m}$. It is the subject of this paper to prove Condition 2 for sufficiently differentiable functions F. A useful application of the results presented here is given by a method for global optimization (Gutmann [2]). Here the global minimum of a continuous function f: $\mathcal{D} \to \mathbb{R}$ is sought, where $\mathcal{D} \subset \mathbb{R}^d$ is compact. If $x_1, ..., x_n \in \mathcal{D}$ and their function values have been calculated, the next point x_{n+1} is determined as follows. Choose an estimate of the global minimum, f^* say. For each $y \in \mathcal{D} \setminus \{x_1, ..., x_n\}$ there exists a radial basis function $s_y \in \mathcal{A}_{\phi, m}$ that interpolates $(x_1, f(x_1))$, ..., $(x_n, f(x_n))$ and (y, f^*). We take the view that the "least bumpy" of these interpolants yields the most promising location for evaluating the objective function. The "bumpiness" of s_y is measured by its semi-norm. This means that x_{n+1} minimizes $\langle s_y, s_y\rangle$, $y \in \mathcal{D} \setminus \{x_1, ..., x_n\}$. A crucial part of the proof of convergence of this method is to show that suitable functions with bounded support satisfy Condition 2. Another use is that the assumptions that guarantee convergence can be weakened if the objective function f itself satisfies Condition 2.

In Section 2 we use Fourier transforms to give a condition that guarantees that a function F is in the native space. Results for particular types of radial basis functions are derived in Section 3, with a special emphasis on functions with bounded support because of their importance for the global optimization method described above.

Multi-index notation will be used in the following sections. For a multi-index $\alpha = (\alpha_1, ..., \alpha_d)^T \in \mathbb{N}^d$, the order $|\alpha|$ and the factorial $\alpha!$ are defined as

$$|\alpha| := \sum_{i=1}^{d} \alpha_i \qquad \text{and} \qquad \alpha! := \prod_{i=1}^{d} \alpha_i!.$$

The power x^α, $x \in \mathbb{R}^d$, and the derivative $D^\alpha u(x)$ for a sufficiently smooth function $u: \mathbb{R}^d \to \mathbb{R}$ are

$$x^\alpha := \prod_{i=1}^{d} x_i^{\alpha_i} \qquad \text{and} \qquad D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}},$$

respectively.

2. A SUFFICIENT CONDITION

For an absolutely integrable function $f: \mathbb{R}^d \to \mathbb{R}$, the Fourier transform $\hat{f}$ is defined as

$$\hat{f}(t) := \int_{\mathbb{R}^d} f(x)\, e^{-ix^T t}\, dx, \qquad t \in \mathbb{R}^d. \tag{2.1}$$

Among the choices of ϕ in (1.2), $\phi(\|.\|)$ is absolutely integrable only in the Gaussian case. However, in all cases, $\phi(\|.\|)$ has a generalized transform $\hat{\phi}(\|t\|)$, $t \in \mathbb{R}^d \backslash \{0\}$. The values of $\hat{\phi}$ are (see Powell [5])

$$\left.\begin{array}{ll} \hat{\phi}(r) = C\, r^{-d-1} & \text{(linear)}, \\ \hat{\phi}(r) = C\, r^{-d-3} & \text{(cubic)}, \\ \hat{\phi}(r) = C\, r^{-d-2} & \text{(thin plate spline)}, \\ \hat{\phi}(r) = C\, (c/r)^{(d+1)/2} K_{(d+1)/2}(cr) & \text{(multiquadric)}, \\ \hat{\phi}(r) = C\, e^{-r^2/4} & \text{(Gaussian)}, \end{array}\right\} r > 0. \tag{2.2}$$

In each case C is a constant that does not depend on t and c, and $r = \|t\|$, $t \in \mathbb{R}^d$. $K_{(d+1)/2}$ is a modified Bessel function that is positive for $r > 0$, and that decays exponentially as r tends to infinity (Abramowitz and Stegun [1]).

To derive a sufficient condition for functions to be in the native space, we use the following criterion that is part of Theorem 8.1 in [8].

PROPOSITION 3. *Let $F: \mathbb{R}^d \to \mathbb{R}$ be continuous, and assume there exists a real number C_F, such that, for any choice of points $x_1, ..., x_n \in \mathbb{R}^d$ and numbers $\lambda_1, ..., \lambda_n \in \mathbb{R}$ that satisfy* (1.5), *the inequality*

$$\left| \sum_{j=1}^{n} \lambda_j F(x_j) \right| \leqslant C_F \left((-1)^{m_0+1} \sum_{i,\, j}^{n} \lambda_i \lambda_j \phi(\|x_i - x_j\|) \right)^{1/2} \tag{2.3}$$

holds. Then F satisfies Condition 2, *thus it is in the native space.*

Proof. Let $x_1, ..., x_n \in \mathbb{R}^d$ satisfy (1.9), and let $s = \sum_{j=1}^n \lambda_j \phi(\|. - x_j\|) + p$ be the unique interpolant to F at these points. Then (2.3) implies

$$\langle s, s \rangle = \left| \sum_{j=1}^n \lambda_j F(x_j) \right| \leqslant C_F \left((-1)^{m_0+1} \sum_{i,j} \lambda_i \lambda_j \phi(\|x_i - x_j\|) \right)^{1/2}$$
$$= C_F \langle s, s \rangle^{1/2}.$$

Thus $\langle s, s \rangle$ is bounded by C_F^2. As C_F does not depend on the choice of the interpolation points, Condition 2 holds. ▮

Further, a lemma is needed that provides an integral representation of the semi-norm of a function in $\mathscr{A}_{\phi, m}$.

LEMMA 4. *Let* ϕ *be any function from* (1.2) *and* $m \geqslant m_0$. *Then for* $s = \sum_{i=1}^n \lambda_i \phi(\|. - x_i\|) \in \mathscr{A}_{\phi, m}$

$$\langle s, s \rangle = \left(\frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} \left| \sum_{j=1}^n \lambda_j e^{i x_j^T t} \right|^2 |\hat{\phi}(\|t\|)| \, dt.$$

Proof. See the proof of Theorem 4.5 in Schaback and Wendland [10]. ▮

The main result of this section is that the function space $\tilde{\mathscr{N}}_\phi$ defined below is contained in the native space.

DEFINITION 5. The linear space $\tilde{\mathscr{N}}_\phi$ is the space of continuous functions $f: \mathbb{R}^d \to \mathbb{R}$ that are absolutely integrable and whose Fourier transform $\hat{f}$ satisfies

$$\int_{\mathbb{R}^d} |\hat{f}(t)|^2 \frac{1}{|\hat{\phi}(\|t\|)|} \, dt < \infty.$$

It is endowed with the inner product

$$(f, g) := \left(\frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} \hat{f}(t) \, \overline{\hat{g}(t)} \, \frac{1}{|\hat{\phi}(\|t\|)|} \, dt$$

that induces the norm

$$(f, f)^{1/2} = \left[\left(\frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} |\hat{f}(t)|^2 \frac{1}{|\hat{\phi}(\|t\|)|} \, dt \right]^{1/2}.$$

Remark 6. The Fourier transform of a function $f \in \tilde{\mathcal{N}}_\phi$ is absolutely integrable which can be shown as follows. If we denote the closed unit ball by B and observe that $\hat{\phi}(\|.\|)$ is in $L^1(\mathbb{R}^d \backslash B)$ in all the cases in (2.2), we find

$$\begin{aligned}\int_{\mathbb{R}^d} |\hat{f}(t)|\, dt &= \int_B |\hat{f}(t)|\, dt + \int_{\mathbb{R}^d \backslash B} \frac{|\hat{f}(t)|}{\sqrt{|\hat{\phi}(\|t\|)|}} \sqrt{|\hat{\phi}(\|t\|)|}\, dt \\ &\leqslant \int_B |\hat{f}(t)|\, dt + \left(\int_{\mathbb{R}^d \backslash B} |\hat{f}(t)|^2 \frac{1}{|\hat{\phi}(\|t\|)|}\, dt \right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{R}^d \backslash B} |\hat{\phi}(\|t\|)|\, dt \right)^{1/2} < \infty.\end{aligned}$$

In particular, the inverse Fourier transform theorem holds, i.e.

$$f(x) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{f}(t)\, e^{it^T x}\, dt, \qquad x \in \mathbb{R}^d.$$

Now we can show that all functions in $\tilde{\mathcal{N}}_\phi$ are in $\mathcal{N}_{\phi, m}$.

THEOREM 7. *Let ϕ be any function from* (1.2) *and $m \geqslant m_0$. Then every function $F \in \tilde{\mathcal{N}}_\phi$ satisfies Condition* 2, *thus it is in $\mathcal{N}_{\phi, m}$.*

Proof. We show that the condition of Proposition 3 holds. Let points $x_1, ..., x_n \in \mathbb{R}^d$ and real numbers $\lambda_1, ..., \lambda_n$ be given that satisfy (1.5). Remark 6 states that the inverse Fourier transform formula for F holds. Thus

$$\begin{aligned}\sum_{j=1}^n \lambda_j F(x_j) &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{F}(t) \left(\sum_{j=1}^n \lambda_j e^{it^T x_j} \right) dt. \\ &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{F}(t) \frac{1}{\sqrt{|\hat{\phi}(\|t\|)|}} \sqrt{|\hat{\phi}(\|t\|)|} \left(\sum_{j=1}^n \lambda_j e^{it^T x_j} \right) dt.\end{aligned}$$

The Cauchy–Schwarz inequality for $L^2(\mathbb{R}^d)$ and Lemma 4 imply

$$\begin{aligned}\left| \sum_{j=1}^n \lambda_j F(x_j) \right| &\leqslant (F, F)^{1/2} \left(\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \left| \sum_{j=1}^n \lambda_j e^{it^T x_j} \right|^2 |\hat{\phi}(\|t\|)|\, dt \right)^{1/2} \\ &= (F, F)^{1/2} \left((-1)^{m_0+1} \sum_{i, j=1}^n \lambda_i \lambda_j \phi(\|x_i - x_j\|) \right)^{1/2}.\end{aligned}$$

By setting $C_F = (F, F)^{1/2}$ the assumption of Proposition 3 is satisfied which proves the theorem. ∎

3. RESULTS FOR PARTICULAR TYPES OF RADIAL BASIS FUNCTIONS

In the linear, cubic and thin plate spline cases, a large class of functions is in the native space. Specifically, let $\kappa = 1$ in the linear case, $\kappa = 2$ in the thin plate spline case and $\kappa = 3$ in the cubic case. Then the following result holds.

THEOREM 8. *Let* $\phi(r) = r$, $\phi(r) = r^2 \log r$ *or* $\phi(r) = r^3$, *and let* $m \geqslant m_0$. *Let* $F \in C^\nu(\mathbb{R}^d)$, *with* $\nu = (d+\kappa)/2$, *if* $d+\kappa$ *is even, or* $\nu = (d+\kappa+1)/2$, *if* $d+\kappa$ *is odd, where* κ *is defined above. Assume that* F *and all derivatives* $D^\alpha F$, $|\alpha| \leqslant \nu$, *are absolutely integrable, and that* $D^\alpha F$ *is in* $L^2(\mathbb{R}^d)$, $|\alpha| = \nu$. *Then* F *is in* $\mathcal{N}_{\phi, m}$.

Proof. It will be proved that F is in the linear space $\tilde{\mathcal{N}}_\phi$. Integration by parts shows that the Fourier transform $\hat{F}$ satisfies

$$\widehat{D^\alpha F}(t) = (it)^\alpha \hat{F}(t), \qquad t \in \mathbb{R}^d, \tag{3.1}$$

for any multi-index α with order $|\alpha| = \nu$. Also, as $D^\alpha F$ is in $L^2(\mathbb{R}^d)$, the Plancherel formula gives

$$\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} |\widehat{D^\alpha F}(t)|^2 \, dt = \int_{\mathbb{R}^d} |D^\alpha F(x)|^2 \, dx < \infty, \qquad |\alpha| = \nu. \tag{3.2}$$

Moreover, expression (2.2) states that $\hat{\phi}$ has the form

$$\hat{\phi}(\|t\|) = \frac{C}{\|t\|^{d+\kappa}}, \qquad t \in \mathbb{R}^d \backslash \{0\},$$

for a constant C, and a generalization of the binomial formula provides

$$\|t\|^{2\nu} = \sum_{|\alpha| = \nu} \frac{\nu!}{\alpha!} t^{2\alpha}, \qquad t \in \mathbb{R}^d.$$

Thus, if $d+\kappa$ is even, the term (F, F) of Definition 5 has the value

$$\begin{aligned}(F, F) &= \left(\frac{1}{2\pi}\right)^d |C|^{-1} \int_{\mathbb{R}^d} |\hat{F}(t)|^2 \, \|t\|^{2\nu} \, dt \\ &= \left(\frac{1}{2\pi}\right)^d |C|^{-1} \sum_{|\alpha|=\nu} \frac{\nu!}{\alpha!} \int_{\mathbb{R}^d} |\hat{F}(t)|^2 \, |t^\alpha|^2 \, dt \\ &= \left(\frac{1}{2\pi}\right)^d |C|^{-1} \sum_{|\alpha|=\nu} \frac{\nu!}{\alpha!} \int_{\mathbb{R}^d} |\widehat{D^\alpha F}(t)|^2 \, dt < \infty.\end{aligned}$$

Analogously, if $d+\kappa$ is odd, one obtains

$$(F, F) = \left(\frac{1}{2\pi}\right)^d |C|^{-1} \sum_{|\alpha|=\nu} \frac{\nu!}{\alpha!} \int_{\mathbb{R}^d} |\widehat{D^\alpha F}(t)|^2 \frac{1}{\|t\|} \, dt. \tag{3.4}$$

For each multi-index α with order ν, (3.1) shows that $\lim_{t \to 0} |\widehat{D^\alpha F}(t)|^2 \, \|t\|^{-1} = 0$. It follows from (3.2) that (F, F) is finite. Thus, in both cases, $F \in \widetilde{\mathcal{N}_\phi}$. Therefore Theorem 7 shows that $F \in \mathcal{N}_{\phi, m}$. ▮

If a smooth function has bounded support, then its derivatives are in $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$. Thus we obtain

Corollary 9. *Let ϕ, m and ν be as in Theorem* 8. *If $F \in C^\nu(\mathbb{R}^d)$ has bounded support, then $F \in \mathcal{N}_{\phi, m}$.*

The corollary includes some well-known special cases including the following one.

Remark 10. Consider natural cubic splines in one dimension, i.e. $d=1$, $\phi(r)=r^3$ and $m=1$. The variational principle (e.g. Powell [4]) states that for $F \in C^2(\mathbb{R})$ with bounded support, the cubic spline interpolant s at finitely many points satisfies

$$\int_{\mathbb{R}} s''(x)^2 \, dx \leqslant \int_{\mathbb{R}} F''(x)^2 \, dx.$$

It can be shown easily that the left hand side equals $12\langle s, s\rangle$. Thus the constant $(1/12) \int F''(x)^2 \, dx$ is a uniform bound on $\langle s, s\rangle$. The existence of such a bound is included in Corollary 9, and the corollary also holds for $m \geqslant 2$.

In the multiquadric and Gaussian cases, however, there is no useful class of functions with bounded support. In these cases, $\hat{\phi}(\|t\|)$ decays exponentially, as $\|t\| \to \infty$. One can show that the only function in $\widetilde{\mathcal{N}_\phi}$ with bounded support is the zero function. This follows from a Paley–Wiener theorem (see Katznelson [3] for a one-dimensional version).

THEOREM 11. *Let $F: \mathbb{R}^d \to \mathbb{R}$ be continuous and compactly supported, and let $e^{\beta \|\cdot\|} \hat{F} \in L^2(\mathbb{R}^d)$ for some $\beta > 0$. Then $F \equiv 0$.*

Proof. The assumption $e^{\beta \|\cdot\|} \hat{F} \in L^2(\mathbb{R}^d)$ shows that $\hat{F}$ is absolutely integrable. Therefore the inverse Fourier transform formula

$$F(x) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{F}(t)\, e^{itx}\, dt, \qquad x \in \mathbb{R}^d, \tag{3.3}$$

holds.

Let $D := \{z \in \mathbb{C} : |\text{Im } z| < \beta\}$. For any arbitrary $v \in \mathbb{R}^d$ with $\|v\| = 1$ define $g : D \to \mathbb{C}$ as

$$g(z) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{F}(t)\, e^{i(zv)^T t}\, dt, \qquad z \in D.$$

We show that g is well-defined and continuous on D. For $z \in D$

$$\begin{aligned} \int_{\mathbb{R}^d} |\hat{F}(t)\, e^{iz(v^T t)}|\, dt &= \int_{\mathbb{R}^d} |\hat{F}(t)\, e^{-\text{Im } z (v^T t)}|\, dt \\ &= \int_{\mathbb{R}^d} |\hat{F}(t)\, e^{\beta \|t\|}|\, |e^{-\beta \|t\|}\, e^{-\text{Im } z (v^T t)}|\, dt \\ &\leqslant \|\hat{F} e^{\beta \|\cdot\|}\|_{L^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} e^{-2(\text{Im } z + \beta)\|t\|}\, dt\right)^{1/2}. \end{aligned}$$

The right hand side is finite, so g is well-defined on D. The continuity of g can be deduced as follows. Fix $z \in D$. Then there is $\alpha \in \mathbb{R}$ such that $|\text{Im } z| < \alpha < \beta$. For $w \in D$ sufficiently close to z, $|\text{Im } w| < \alpha$, which provides the bound

$$|\hat{F}(t)\, e^{iw(v^T t)}| = |\hat{F}(t)|\, e^{-(v^T t)\, \text{Im } w} \leqslant |\hat{F}(t)|\, e^{\alpha \|t\|}, \qquad t \in \mathbb{R}^d.$$

The function on the right hand side is integrable, because a similar argument as above gives

$$\begin{aligned} \int_{\mathbb{R}^d} |\hat{F}(t)|\, e^{\alpha \|t\|}\, dt &= \int_{\mathbb{R}^d} |\hat{F}(t)\, e^{\beta \|t\|}|\, e^{-(\beta - \alpha)\|t\|}\, dt \\ &\leqslant \|\hat{F} e^{\beta \|\cdot\|}\|_{L^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} e^{-2(\beta - \alpha)\|t\|}\, dt\right)^{1/2} < \infty. \end{aligned}$$

Therefore, as w tends to z, the dominated convergence theorem implies

$$g(z)=\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \lim_{w\to z} \hat{F}(t)\, e^{iw(v^T t)}\, dt$$
$$= \lim_{w\to z} \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{F}(t)\, e^{iw(v^T t)}\, dt = \lim_{w\to z} g(w).$$

Thus g is continuous on D.

Further, Morera's Theorem (see e.g. Rudin [6]) shows that g is analytic on D. Let T be any triangle in the interior of D, and denote its boundary by ∂T. Then, by Fubini's Theorem,

$$\int_{\partial T} g(z)\, dz = \left(\frac{1}{2\pi}\right)^d \int_{\partial T} \int_{\mathbb{R}^d} \hat{F}(t)\, e^{i(zv)^T t}\, dt\, dz$$
$$= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{F}(t) \int_{\partial T} e^{i(v^T t)\, z}\, dz\, dt = 0,$$

because $e^{i(v^T t)\, z}$ is analytic for every $t \in \mathbb{R}^d$. As T is arbitrary, g is analytic.

Now choose $\tau^* \in \mathbb{R}$ such that $\tau^* v$ is outside the support of F. Because g and $F(.\, v)$ coincide on the real line, g and all its derivatives vanish at τ^*. Therefore the Taylor expansion of g around τ^* yields that $g \equiv 0$ on D. Thus $F(\tau v) = 0$ for all $\tau \in \mathbb{R}$. Because v has been chosen arbitrarily, this proves the theorem. ∎

It follows that in the multiquadric and Gaussian cases $\mathcal{N}_\phi$ contains no nonzero function with compact support.

COROLLARY 12. *Let* $\phi(r) = \sqrt{r^2 + c^2}$ *or* $\phi(r) = e^{-r^2}$. *If* $F \in \tilde{\mathcal{N}}_\phi$ *has bounded support, then* $F \equiv 0$.

Proof. In both cases considered here, $\hat{\phi}(\|.\|)$ decays exponentially, i.e. there exist $\beta > 0$ and $K > 0$ such that

$$|\hat{\phi}(\|t\|)| \leqslant K e^{-2\beta \|t\|}, \qquad t \in \mathbb{R}^d \backslash B, \tag{3.4}$$

where B is the closed unit ball. Now $F \in \tilde{\mathcal{N}}_\phi$ provides

$$\int_{\mathbb{R}^d \backslash B} |\hat{F}(t)|^2\, e^{2\beta \|t\|}\, dt \leqslant K \int_{\mathbb{R}^d \backslash B} |\hat{F}(t)|^2 \frac{1}{|\hat{\phi}(\|t\|)|}\, dt < \infty.$$

It follows that $e^{\beta \|.\|} \hat{F} \in L^2(\mathbb{R}^d)$. Theorem 11 now implies the required result. ∎

4. CONCLUSIONS

We have introduced a new technique to prove Condition 2 for sufficiently smooth functions with bounded support. It provides an extension of well-known results in the linear, cubic and thin plate spline cases. In particular, the choice of m is less restricted than before.

The situation in the multiquadric and Gaussian cases, however, is disappointing. The exponential decay of $\hat{\phi}$ prevents a uniform bound on the semi-norms when F has bounded support. As mentioned in the introduction, the application to global optimization we have in mind requires Condition 2 for such functions. Thus the presented approach is not useful in these cases. In the other cases the question arises whether there exist functions that are not in $\tilde{\mathcal{N}}_\phi$ but still satisfy Condition 2. In particular, it is interesting to investigate how the semi-norm behaves when F is less differentiable than required by Theorem 8.

ACKNOWLEDGMENTS

First, I thank two anonymous referees, one who pointed out Proposition 3 which simplified the argument. Also, I am very grateful to my supervisor M.J.D. Powell, who was always prepared to help over difficulties, as well as to Brad Baxter and Michael Johnson who gave valuable advice. Furthermore, this research has been supported by the German Academic Exchange Service through a doctoral scholarship (HSP III) and by the Engineering and Physical Sciences Research Council.

REFERENCES

1. M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions," Applied Mathematics Series (National Bureau of Standards), Vol. 55, United States Department of Commerce, 1964.
2. H.-M. Gutmann, A radial basis function method for global optimization, *J. Global Optim.* **19**, No. 3 (2001), 201–227.
3. Y. Katznelson, "An Introduction to Harmonic Analysis," Dover Publications, New York, 1976.
4. M. J. D. Powell, "Approximation Theory and Methods," Cambridge University Press, Cambridge, UK, 1981.
5. M. J. D. Powell, The theory of radial basis function approximation in 1990, *in* "Advances in Numerical Analysis, Volume 2: Wavelets, Subdivision Algorithms and Radial Basis Functions" (W. A. Light, Ed.), pp. 105–210, Oxford University Press, Oxford, 1992.
6. W. Rudin, "Real and Complex Analysis," Mc Graw-Hill, Singapore, 1987.
7. R. Schaback, Comparison of radial basis function interpolants, *in* "Multivariate Approximations: From CAGD to Wavelets" (K. Jetter and F. Utreras, Eds.), pp. 293–305, World Scientific, Singapore, 1993.

8. R. Schaback, Native Hilbert spaces for radial basis functions I, *in* “New Developments in Approximation Theory” (M. W. Müller, M. D. Buhmann, D. H. Mache, and M. Felten, Eds.), pp. 255–282, Birkhäuser Verlag, Basel, 1999.
9. R. Schaback and H. Wendland, Inverse and saturation theorems for radial basis function interpolation, preprint, Universität Göttingen, 1998.
10. R. Schaback and H. Wendland, Characterization and construction of radial basis functions, preprint, Universität Göttingen, 2000.